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N-Fractional Calculus of Some Functions Which Include A Root Sign(Study on Geometric Univalent Function Theory)

AUTHOR(S):

Nishimoto, Katsuyuki

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N- Fractional Calculus of Some Functions Which Include A Root Sign

Katsuyuki Nishimoto

Institute for Applied Mathematics, Descartes Press Co.

2- 13- 10 Kaguike, Koriyama, 963 - 8833, JAPAN

Fax : + 81 - 24 - 922 - 7596

Abstract

In a previous article of the author, N- fractional calculus of some composite algebraic functions are derived. Applying this fresh results, N- fractional calculus of functions

$$\frac{1}{\sqrt[m]{(z-b)^m - c}} \quad \text{and} \quad \sqrt[m]{(z-b)^m - c} \quad (m \in \mathbb{Z}^+)$$

are reported in this paper. That is , we have the below, for example.

$$(i) \quad \left(\frac{1}{\sqrt[m]{(z-b)^m - c}} \right)_\gamma = e^{-i\pi\gamma} (z-b)^{-1-\gamma} \\ \times \sum_{k=0}^{\infty} \frac{[1/m]_k \Gamma(mk+1+\gamma)}{k! \Gamma(mk+1)} \left(\frac{c}{(z-b)^m} \right)^k \quad (|\Gamma(mk+1+\gamma)| < \infty)$$

and

$$(ii) \quad \left(\sqrt[m]{(z-b)^m - c} \right)_\gamma = e^{-i\pi\gamma} (z-b)^{1-\gamma} \\ \times \sum_{k=0}^{\infty} \frac{[-1/m]_k \Gamma(mk-1+\gamma)}{k! \Gamma(mk-1)} \left(\frac{c}{(z-b)^m} \right)^k \quad (|\Gamma(mk-1+\gamma)| < \infty)$$

where

$$|c/(z-b)^m| < 1, \quad m \in \mathbb{Z}^+,$$

and

$$[\lambda]_k = \lambda(\lambda+1) \cdots (\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \quad \text{with } [\lambda]_0 = 1, \\ (\text{Notation of Pochhammer}).$$

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\text{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\text{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_v = (f)_v = {}_C(f)_v = \frac{\Gamma(v+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{v+1}} d\xi \quad (v \notin \mathbb{Z}), \quad (1)$$

$$(f)_{-m} = \lim_{v \rightarrow -m} (f)_v \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\xi-z) \leq \pi$ for C_- , $0 \leq \arg(\xi-z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in C$, $v \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_v$ is the fractional differintegration of arbitrary order v (derivatives of order v for $v > 0$, and integrals of order $-v$ for $v < 0$), with respect to z , of the function f , if $|(f)_v| < \infty$.

(II) On the fractional calculus operator N^v [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^v be

$$N^v = \left(\frac{\Gamma(v+1)}{2\pi i} \int_C \frac{d\xi}{(\xi-z)^{v+1}} \right) \quad (v \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3)$$

$$\text{with} \quad N^{-m} = \lim_{v \rightarrow -m} N^v \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation \circ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^v\} = \{N^v | v \in \mathbb{R}\} \quad (6)$$

is an Abelian product group (having continuous index v) which has the inverse transform operator $(N^v)^{-1} = N^{-v}$ to the fractional calculus operator N^v , for the function f such that $f \in F = \{f; 0 \neq |f_v| < \infty, v \in \mathbb{R}\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < v < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^v\}$ " is an " Action product group which has continuous index v " for the set of F . (F.O.G. ; Fractional calculus operator group) [3]

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \mathbb{R}). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S). \quad (8)$$

(III) **Lemma.** We have [1]

$$(i) \quad ((z-c)^b)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-b)}{\Gamma(-b)} (z-c)^{b-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-b)}{\Gamma(-b)} \right| < \infty \right),$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty),$$

$$(iii) \quad ((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty),$$

where $z-c \neq 0$ for (i) and $z-c \neq 0, 1$ for (ii), (iii),

$$(iv) \quad (u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \begin{pmatrix} u = u(z), \\ v = v(z) \end{pmatrix}.$$

§ 1. Preliminary

The Theorem below is reported by the author already (cf. J. Frac. Calc. Vol. 29, May (2006), pp.35 - 44.) . [12]

Theorem D. We have

$$(i) \quad \begin{aligned} &(((z-b)^\beta - c)^\alpha)_\gamma = e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma} \\ &\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta} \right)^k \\ &\left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right) \end{aligned} \quad (1)$$

and

$$(ii) \quad \begin{aligned} &(((z-b)^\beta - c)^\alpha)_n = (-1)^n (z-b)^{\alpha\beta-n} \\ &\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^\beta} \right)^k \quad (n \in \mathbb{Z}_0^+) \end{aligned} \quad (2)$$

where

$$\left| \frac{c}{(z-b)^\beta} \right| < 1,$$

and

$[\lambda]_k = \lambda(\lambda + 1) \cdots (\lambda + k - 1) = \Gamma(\lambda + k) / \Gamma(\lambda)$ with $[\lambda]_0 = 1$,
(Notation of Pochhammer).

§ 2. N- Fractional Calculus of Functions

$$\frac{1}{\sqrt[m]{(z-b)^m - c}} \quad \text{and} \quad \sqrt[m]{(z-b)^m - c}, \quad (m \in \mathbb{Z}^+)$$

Applying Theorem D in § 1. Preliminary we obtain the following theorems.

Theorem 1. We have

$$(i) \quad \left(\frac{1}{\sqrt[m]{(z-b)^m - c}} \right)_\gamma = e^{-ix\gamma} (z-b)^{-1-\gamma} \\ \times \sum_{k=0}^{\infty} \frac{[1/m]_k \Gamma(mk + 1 + \gamma)}{k! \Gamma(mk + 1)} \left(\frac{c}{(z-b)^m} \right)^k \quad (|\Gamma(mk + 1 + \gamma)| < \infty) \quad (1)$$

and

$$(ii) \quad \left(\frac{1}{\sqrt[m]{(z-b)^m - c}} \right)_n = (-1)^n (z-b)^{-1-n} \\ \times \sum_{k=0}^{\infty} \frac{[1/m]_k [mk + 1]_n}{k!} \left(\frac{c}{(z-b)^m} \right)^k \quad (n \in \mathbb{Z}_0^+) \quad (2)$$

where

$$|c/(z-b)^m| < 1, \quad m \in \mathbb{Z}^+.$$

Proof of (i). We have

$$\left(\frac{1}{\sqrt[m]{(z-b)^m - c}} \right)_\gamma = \left(((z-b)^m - c)^{-1/m} \right)_\gamma, \quad (3)$$

therefore, setting $\beta = m$ and $\alpha = -1/m$ in Theorem D. (i), we obtain (1) clearly, under the conditions.

Proof of (ii). Set $\gamma = n$ in (1), we have then (2), since

$$\frac{\Gamma(mk + 1 + n)}{\Gamma(mk + 1)} = [mk + 1]_n. \quad (4)$$

Corollary 1. *We have*

$$(i) \quad \left(\frac{1}{\sqrt{(z-b)^2 - c}} \right)_\gamma = e^{-i\pi\gamma} (z-b)^{-1-\gamma} \\ \times \sum_{k=0}^{\infty} \frac{[1/2]_k \Gamma(2k+1+\gamma)}{k! \Gamma(2k+1)} \left(\frac{c}{(z-b)^2} \right)^k \quad (|\Gamma(2k+1+\gamma)| < \infty)$$

(5)

and

$$(ii) \quad \left(\frac{1}{\sqrt{(z-b)^2 - c}} \right)_n = (-1)^n (z-b)^{-1-n} \\ \times \sum_{k=0}^{\infty} \frac{[1/2]_k [2k+1]_n}{k!} \left(\frac{c}{(z-b)^2} \right)^k \quad (n \in \mathbb{Z}_0^+) \quad (6)$$

where

$$|c/(z-b)^2| < 1.$$

Proof. Set $m = 2$ in Theorem 1.

Corollary 2. *We have*

$$(i) \quad \left(\frac{1}{\sqrt{z^2 - 2bz + p}} \right)_\gamma = e^{-i\pi\gamma} (z-b)^{-1-\gamma} \\ \times \sum_{k=0}^{\infty} \frac{[1/2]_k \Gamma(2k+1+\gamma)}{k! \Gamma(2k+1)} \left(\frac{b^2 - p}{(z-b)^2} \right)^k \quad (|\Gamma(2k+1+\gamma)| < \infty)$$

(7)

and

$$(ii) \quad \left(\frac{1}{\sqrt{z^2 - 2bz + p}} \right)_n = (-1)^n (z-b)^{-1-n} \\ \times \sum_{k=0}^{\infty} \frac{[1/2]_k [2k+1]_n}{k!} \left(\frac{b^2 - p}{(z-b)^2} \right)^k \quad (n \in \mathbb{Z}_0^+) \quad (8)$$

where

$$|(b^2 - p)/(z-b)^2| < 1.$$

Proof of (i). We have

$$z^2 - 2bz + p = (z - b)^2 - c \quad (c = b^2 - p), \quad (9)$$

hence

$$\left(\frac{1}{\sqrt{z^2 - 2bz + p}} \right)_\gamma = \left(\frac{1}{\sqrt{(z - b)^2 - c}} \right)_\gamma. \quad (10)$$

Therefore, we obtain (7) from (10) and (5), under the conditions.

Proof of (ii). Set $\gamma = n$ in (7), we have then (8) clearly.

Theorem 2. We have

$$(i) \quad \left({}^m\sqrt{(z - b)^m - c} \right)_\gamma = e^{-i\pi\gamma} (z - b)^{1-\gamma} \times \sum_{k=0}^{\infty} \frac{[-1/m]_k \Gamma(mk - 1 + \gamma)}{k! \Gamma(mk - 1)} \left(\frac{c}{(z - b)^m} \right)^k \left(\left| \frac{\Gamma(mk - 1 + \gamma)}{\Gamma(mk - 1)} \right| < \infty \right) \quad (11)$$

and

$$(ii) \quad \left({}^m\sqrt{(z - b)^m - c} \right)_n = (-1)^n (z - b)^{1-n} \times \sum_{k=0}^{\infty} \frac{[-1/m]_k [mk - 1]_n}{k!} \left(\frac{c}{(z - b)^m} \right)^k \quad (n \in \mathbb{Z}_0^+) \quad (12)$$

where

$$|c/(z - b)^m| < 1, \quad m \in \mathbb{Z}^+.$$

Proof of (i). We have

$$\left({}^m\sqrt{(z - b)^m - c} \right)_\gamma = \left(((z - b)^m - c)^{1/m} \right)_\gamma, \quad (13)$$

therefore, setting $\beta = m$ and $\alpha = 1/m$ in Theorem D. (i), we obtain (11) clearly, under the conditions.

Proof of (ii). Set $\gamma = n$ in (11).

Corollary 3. We have

$$(i) \quad \left(\sqrt{(z - b)^2 - c} \right)_\gamma = e^{-i\pi\gamma} (z - b)^{1-\gamma} \times \sum_{k=0}^{\infty} \frac{[-1/2]_k \Gamma(2k - 1 + \gamma)}{k! \Gamma(2k - 1)} \left(\frac{c}{(z - b)^2} \right)^k \left(\left| \frac{\Gamma(2k - 1 + \gamma)}{\Gamma(2k - 1)} \right| < \infty \right) \quad (14)$$

and

$$(ii) \quad \left(\sqrt{(z-b)^2 - c} \right)_n = (-1)^n (z-b)^{1-n} \times \sum_{k=0}^{\infty} \frac{[-1/2]_k [2k-1]_n}{k!} \left(\frac{c}{(z-b)^2} \right)^k \quad (n \in \mathbb{Z}_0^+) \quad (15)$$

where

$$|c/(z-b)^2| < 1.$$

Proof. Set $m = 2$ in Theorem 2.

Corollary 4. We have

$$(i) \quad \left(\sqrt{z^2 - 2bz + p} \right)_\gamma = e^{-i\pi\gamma} (z-b)^{1-\gamma} \times \sum_{k=0}^{\infty} \frac{[-1/2]_k \Gamma(2k-1+\gamma)}{k! \Gamma(2k-1)} \left(\frac{b^2 - p}{(z-b)^2} \right)^k \quad \left(\left| \frac{\Gamma(2k-1+\gamma)}{\Gamma(2k-1)} \right| < \infty \right) \quad (16)$$

and

$$(ii) \quad \left(\sqrt{z^2 - 2bz + p} \right)_n = (-1)^n (z-b)^{1-n} \times \sum_{k=0}^{\infty} \frac{[-1/2]_k [2k-1]_n}{k!} \left(\frac{b^2 - p}{(z-b)^2} \right)^k \quad (n \in \mathbb{Z}_0^+) \quad (17)$$

where

$$|(b^2 - p)/(z-b)^2| < 1.$$

Proof of (i). We have (9), hence

$$\left(\sqrt{z^2 - 2bz + p} \right)_\gamma = \left(\sqrt{(z-b)^2 - c} \right)_\gamma. \quad (18)$$

Therefore, we obtain (16) from (18) and (14), under the conditions.

Proof of (ii). Set $\gamma = n$ in (16).

§ 3. Special Cases of Corollaries 1. (ii) and 3. (ii)

[I] Special case of Corollary 1. (ii) (for $n = 0, 1, 2$)

1.) When $n = 0$ we have

$$\left(\frac{1}{\sqrt{(z-b)^2 - c}} \right)_0 = (z-b)^{-1} \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} \left(\frac{c}{(z-b)^2} \right)^k \quad (1)$$

$$= \frac{1}{z-b} \left(1 - \frac{c}{(z-b)^2} \right)^{-1/2} \quad (2)$$

$$= \left(\frac{1}{(z-b)^2 - c} \right)^{1/2} \quad (3)$$

from § 2. (6).

2.) When $n = 1$ we have

$$\left(\frac{1}{\sqrt{(z-b)^2 - c}} \right)_1 = -(z-b)^{-2} \sum_{k=0}^{\infty} \frac{[1/2]_k [2k+1]_1}{k!} \left(\frac{c}{(z-b)^2} \right)^k \quad (4)$$

$$= -(z-b)^{-2} \sum_{k=0}^{\infty} \frac{[1/2]_k (2k+1)}{k!} T^k \quad \left(T = \frac{c}{(z-b)^2} \right) \quad (5)$$

$$= -2(z-b)^{-2} \sum_{k=0}^{\infty} \frac{[1/2]_k k}{k!} T^k - (z-b)^{-2} \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} T^k \quad (6)$$

$$= -2(z-b)^{-2} \cdot \frac{T}{2} (1-T)^{-3/2} - (z-b)^{-2} (1-T)^{-1/2} \quad (7)$$

$$= -(z-b)^{-2} (1-T)^{-3/2} \quad (8)$$

$$= -(z-b) ((z-b)^2 - c)^{-3/2} \quad (9)$$

from § 2. (6). (see Appendix I)

3.) When $n = 2$ we have

$$\left(\frac{1}{\sqrt{(z-b)^2 - c}} \right)_2 = (z-b)^{-3} \sum_{k=0}^{\infty} \frac{[1/2]_k [2k+1]_2}{k!} \left(\frac{c}{(z-b)^2} \right)^k \quad (10)$$

$$= (z-b)^{-3} \sum_{k=0}^{\infty} \frac{[1/2]_k (2k+1)[2k+2]_1}{k!} \left(\frac{c}{(z-b)^2} \right)^k \quad (11)$$

$$= (z-b)^{-3} \left\{ 2 \sum_{k=0}^{\infty} \frac{[1/2]_k k [2k+2]_1}{k!} T^k + \sum_{k=0}^{\infty} \frac{[1/2]_k [2k+2]_1}{k!} T^k \right\} \quad (12)$$

$$= (z-b)^{-3} \left\{ \frac{4T - T^2}{(1-T)^{5/2}} + \frac{2-T}{(1-T)^{3/2}} \right\} \quad (13)$$

$$= (z-b)^{-3} \frac{1}{(1-T)^{3/2}} \left(\frac{T+2}{1-T} \right) \quad (14)$$

$$= ((z-b)^2 - c)^{-5/2} (2(z-b)^2 + c), \quad (15)$$

from § 2. (6). (see Appendix II and III).

The results (9) and (15) coincide with the ones obtained by the classical calculations

$$\frac{d}{dz} \left(\frac{1}{\sqrt{(z-b)^2 - c}} \right) \quad \text{and} \quad \frac{d^2}{dz^2} \left(\frac{1}{\sqrt{(z-b)^2 - c}} \right), \quad (16)$$

respectively. And so on.

Therefore we can see that the presentation of Corollary 1. (ii) holds true for $n \in \mathbb{Z}_0^+$.

[II] Special case of Corollary 3. (ii) (for $n = 0, 1, 2$)

1.) When $n = 0$ we have

$$\left(\sqrt{(z-b)^2 - c} \right)_0 = (z-b) \sum_{k=0}^{\infty} \frac{[-1/2]_k}{k!} \left(\frac{c}{(z-b)^2} \right)^k \quad (17)$$

$$= (z-b) \left(1 - \frac{c}{(z-b)^2} \right)^{1/2} \quad (18)$$

$$= ((z-b)^2 - c)^{1/2} \quad (19)$$

from § 2. (15).

2.) When $n = 1$ we have

$$\left(\sqrt{(z-b)^2 - c} \right)_1 = - \sum_{k=0}^{\infty} \frac{[-1/2]_k [2k-1]_1}{k!} \left(\frac{c}{(z-b)^2} \right)^k \quad (20)$$

$$= - \sum_{k=0}^{\infty} \frac{[-1/2]_k (2k-1)}{k!} T^k \quad \left(T = \frac{c}{(z-b)^2} \right) \quad (21)$$

$$= -2 \sum_{k=0}^{\infty} \frac{[-1/2]_k k}{k!} T^k + \sum_{k=0}^{\infty} \frac{[-1/2]_k}{k!} T^k \quad (22)$$

$$= -T (1-T)^{-1/2} + (1-T)^{1/2} \quad (23)$$

$$= (1-T)^{-1/2} \quad (24)$$

$$= (z-b) ((z-b)^2 - c)^{-1/2}, \quad (25)$$

from § 2. (6). (see Appendix IV.)

3.) When $n=2$ we have

$$\left(\sqrt{(z-b)^2 - c} \right)_2 = (z-b)^{-1} \sum_{k=0}^{\infty} \frac{[-1/2]_k [2k-1]_2}{k!} \left(\frac{c}{(z-b)^2} \right)^k \quad (26)$$

$$= (z-b)^{-1} \sum_{k=0}^{\infty} \frac{[-1/2]_k (2k-1)[2k]_1}{k!} T^k \quad (27)$$

$$= (z-b)^{-1} \left\{ 2 \sum_{k=0}^{\infty} \frac{[-1/2]_k k [2k]_1}{k!} T^k - \sum_{k=0}^{\infty} \frac{[-1/2]_k [2k]_1}{k!} T^k \right\} \quad (28)$$

$$= (z-b)^{-1} \left\{ T(T-2)(1-T)^{-3/2} + T(1-T)^{-1/2} \right\} \quad (29)$$

$$= - (z-b)^{-1} T(1-T)^{-3/2} \quad (30)$$

$$= - \frac{c}{((z-b)^2 - c)^{3/2}}, \quad (31)$$

from § 2. (6). (see Appendix V and VI).

The results (25) and (31) coincide with the ones obtained by the classical calculations

$$\frac{d}{dz} \left(\sqrt{(z-b)^2 - c} \right) \quad \text{and} \quad \frac{d^2}{dz^2} \left(\sqrt{(z-b)^2 - c} \right),$$

respectively, again. And so on.

Therefore we can see that the presentation of Corollary 3. (ii) holds true for $n \in \mathbb{Z}_0^+$.

Appendix

I. We have

$$\sum_{k=0}^{\infty} \frac{[1/2]_k k}{k!} T^k = \sum_{k=1}^{\infty} \frac{[1/2]_k}{(k-1)!} T^k = T \sum_{k=0}^{\infty} \frac{[1/2]_{k+1}}{k!} T^k \quad (1)$$

$$= \frac{1}{2} T \sum_{k=0}^{\infty} \frac{[3/2]_k}{k!} T^k = \frac{1}{2} T (1-T)^{-3/2} \quad (2)$$

II.

$$\sum_{k=0}^{\infty} \frac{[1/2]_k k [2k+2]_1}{k!} T^k = \sum_{k=1}^{\infty} \frac{[1/2]_k [2k+2]_1}{(k-1)!} T^k \quad (3)$$

$$= T \sum_{k=0}^{\infty} \frac{[1/2]_{k+1} [2k+4]_1}{k!} T^k = \frac{1}{2} T \sum_{k=0}^{\infty} \frac{[3/2]_k (2k+4)}{k!} T^k \quad (4)$$

$$= T \sum_{k=0}^{\infty} \frac{[3/2]_k k}{k!} T^k + 2 T \sum_{k=0}^{\infty} \frac{[3/2]_k}{k!} T^k \quad (5)$$

$$= T^2 \sum_{k=0}^{\infty} \frac{[3/2]_{k+1}}{k!} T^k + 2 T (1-T)^{-3/2} \quad (6)$$

$$= \frac{3}{2} T^2 \sum_{k=0}^{\infty} \frac{[5/2]_k}{k!} T^k + 2 T (1-T)^{-3/2} \quad (7)$$

$$= \frac{3}{2} T^2 (1-T)^{-5/2} + 2 T (1-T)^{-3/2} \quad (8)$$

$$= \frac{4T - T^2}{2(1-T)^{5/2}} \quad (9)$$

III.

$$\sum_{k=0}^{\infty} \frac{[1/2]_k [2k+2]_1}{k!} T^k = \sum_{k=1}^{\infty} \frac{[1/2]_k (2k+2)}{k!} T^k \quad (10)$$

$$= 2 \sum_{k=0}^{\infty} \frac{[1/2]_k k}{k!} T^k + 2 \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} T^k \quad (11)$$

$$= T (1-T)^{-3/2} + 2 (1-T)^{-1/2} \quad (12)$$

$$= \frac{2-T}{(1-T)^{3/2}} \quad (13)$$

$$\text{IV.} \quad \sum_{k=0}^{\infty} \frac{[-1/2]_k k}{k!} T^k = \sum_{k=1}^{\infty} \frac{[-1/2]_k}{(k-1)!} T^k \quad (14)$$

$$= T \sum_{k=0}^{\infty} \frac{[-1/2]_{k+1}}{k!} T^k = -\frac{1}{2} T \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} T^k \quad (15)$$

$$= -\frac{1}{2} T (1-T)^{-1/2}. \quad (16)$$

$$\text{V.} \quad \sum_{k=0}^{\infty} \frac{[-1/2]_k k [2k]_1}{k!} T^k = \sum_{k=1}^{\infty} \frac{[-1/2]_k [2k]_1}{(k-1)!} T^k \quad (17)$$

$$= T \sum_{k=0}^{\infty} \frac{[-1/2]_{k+1} [2k+2]_1}{k!} T^k = -\frac{T}{2} \sum_{k=0}^{\infty} \frac{[1/2]_k (2k+2)}{k!} T^k \quad (18)$$

$$= -T \sum_{k=0}^{\infty} \frac{[1/2]_k k}{k!} T^k - T \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} T^k \quad (19)$$

$$= -\frac{1}{2} T^2 (1-T)^{-3/2} - T (1-T)^{-1/2} \quad (20)$$

$$= -\frac{1}{2} T(T-2)(1-T)^{-3/2}. \quad (21)$$

$$\text{VI.} \quad \sum_{k=0}^{\infty} \frac{[-1/2]_k [2k]_1}{k!} T^k = \sum_{k=0}^{\infty} \frac{[-1/2]_k (2k)}{(k-1)!} T^k \quad (22)$$

$$= 2 \sum_{k=1}^{\infty} \frac{[-1/2]_k}{(k-1)!} T^k = 2T \sum_{k=0}^{\infty} \frac{[-1/2]_{k+1}}{k!} T^k \quad (23)$$

$$= -T \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} T^k \quad (24)$$

$$= -T (1-T)^{-1/2}. \quad (25)$$

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Katsuyuki Nishimoto
 Institute for Applied Mathematics
 Descartes Press Co.
 2 - 13 - 10 Kaguike, Koriyama
 963 - 8833 Japan